Accurate FDTD wavelet–Galerkin representation of field singularities near conductive wedges

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Abstract: A novel hybrid technique for the precise representation of field singlarities generated by hard to model geometrical peculiarities, such as arbitrarily angled conductive wedges, is presented. Its primary concept lies in the combined implementation of the FDTD method and the wavelet (Daubechies' basis)–Galerkin formulation in different, distinct areas of the computational domain. The robustness and simplicity of the former, used in regions of smooth field variations, and the ability of the latter, utilised near discontinuities, to efficiently simulate highly varying phenomena, allow the precise treatment of sharp wedges. Hence, the proposed algorithm yields sufficiently accurate coarse grid results and short time-advancing intervals, as the numerical verification reveals.

1 Introduction

The accurate representation of electromagnetic field spatial disturbances due to sharp conductive wedges has been a point of intensive theoretical and numerical research for diverse microwave areas such as waveguiding, scattering and optics. Since the initial discovery of this problem, various techniques have been proposed, based either on denser FDTD/FEM discretisations of the domain [1] (global solutions), or on the development of hybrid schemes aiming at the reduction of the overall computational burden [2, 3] (local solutions). However, numerical simulations indicate that the former algorithms utilise a great deal of CPU and memory resources, while the latter, despite being very efficient, turn out to be mathematically more complex, contrary to the simplicity characterising the standard FDTD or FEM codes.

The objective of this paper is to introduce a new hybrid FDTD wavelet–Galerkin technique, which can effectively approximate highly-varying local field disturbances in the vicinity of waveguide discontinuities (Fig. 1). By properly determining the number and lattice location of field components requiring special treatment, a wavelet-based formulation is locally applied, whereas the conventional FDTD algorithm handles the rest of the computational domain. Moreover, taking advantage of the Daubechies' functions' [4] shifted interpolation property, and maintaining the FDTD temporal and spatial increments uniformly unchanged, the proposed scheme achieves exact simulation of the aforementioned phenomenon without the need for transition conditions at the interface of the two methods. Therefore, the hybrid technique preserves its simplicity and its explicit character, unlike other localised approaches, which, based on a series expansion of field components near singnlarities, require either matrix inversion in each time step [2], or complexity in programming [3]. Consequently, fairly accurate numerical field representations by means of coarse grids and significantly reduced overall time factors are attained for complicated waveguide structures, where existing techniques do not give satisfying performance.

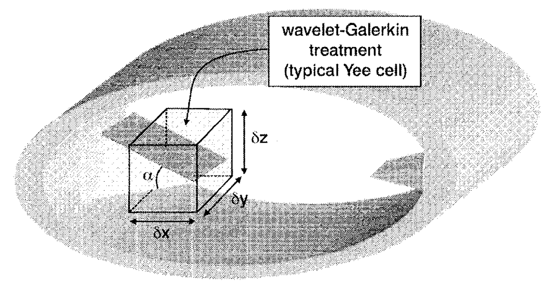

Fig. 1 *General 3D curvilinear waveguide structure with singularity regions*

2 The wavelet–Galerkin formulation

The wavelet–Galerkin method is classified among the few schemes that can combine computational electromagnetics with the theory of wavelets. Since its initial advent, various efficient techniques have incorporarated its notable characteristics, such as the wavelet-based method of moments [5, 6], the Battle–Lemarie MRTD [7], the Daubechies wavelet-based [8, 9], and the Haar wavelet-based [10] methods. Their fundamental concept stems from the multiresolution analysis, which allows the expansion of each field component into a series of orthonormal basis functions:

$$F(x,y,z,t) = \sum_i \sum_j \sum_k \sum_n F^n_{ijk} \phi_i(x)\phi_j(y)\phi_k(z)h_n(t) \quad (1)$$

where F represents any electric or magnetic field component, $h_n(t)$ is the well known Haar scaling function, and $\phi_\zeta(\xi) - \zeta = i, j, k$ ($\xi = x, y, z$) denotes an appropriate scaling function, herein efficiently selected to be Daubechies'. According to eqn. 1, the computation of $F(i\delta x, j\delta y, k\delta z, n\delta t)$ theoretically requires the values of all $F_{ijk}{}^n$ everywhere

IEE Proceedings online no. 20010422
DOI: 10.1049/ip-map:20010422
Paper first received 22nd June 2000 and in revised form 20th March 2001
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in the lattice, thus complicating the entire procedure. However, due to the Daubechies' functions' properties (compact support and shifted interpolation), only the $F_{ijk}{}^{n}$ term at the (i, j, k) node is adequate for the consistent representation of F, without loss of accuracy. Namely, the value of $F_{ijk}{}^{n}$ equals to the one of $F(i\delta x, j\delta y, k\delta z, n\delta t)$, therefore the series coefficients are located at the same positions as the FDTD field components, something very important for the implementation of the proposed technique.

According to the Galerkin weighted residuals procedure, eqn. 1 is substituted into Maxwell's curl equations, thus forming a set of six equations containing scaling functions $h(t)$, $\phi(x)$, $\phi(y)$ and $\phi(z)$, as well as their derivatives with respect to time t, and directions x, y and z, respectively. The resulted relations are weighted by proper basis functions, yielding inner products of the form

$$\langle \phi_\zeta(\xi), \phi_{\zeta'}(\xi) \rangle = \delta_{\zeta,\zeta'} \delta\xi \tag{2}$$

and

$$\left\langle \phi_\zeta(\xi), \frac{d\phi_{\zeta'}(\xi)}{d\xi} \right\rangle = \sum_{ps} C_{ps} \delta_{\zeta,\zeta'+ps} \delta\xi \tag{3}$$

where $\delta_{\zeta\zeta'}$ is Kronecker's delta. The coefficients C_{ps}, implemented for the representation of the operator $d/d\xi$ by an orthonormal base of compactly supported wavelets, depend only on the selected scaling function $\delta(\xi)$, and are calculated by the relation [11]

$$C_{ps} = \int_{-\infty}^{+\infty} \phi(\xi + ps) \frac{d\phi(\xi - 0.5)}{d\xi} d\xi \tag{4}$$

Table 1 presents the coefficients C_{ps} computed numerically for Daubechies' scaling functions (DBN), with various numbers of wavelet vanishing moments N.

Table 1: C_{ps} scaling function coefficients

ps	DB2	DB3	DB5
0	1.233348	1.291646	1.3160011
1	−0.093853	−0.137122	−0.1642694
2	0.010618	0.028742	0.0503897
3		−0.003466	−0.0135287
4		7.98963e-6	0.0022954
5			−1.48145e-4
6			−8.60219e-6
7			5.73355e-8
8			2.85768e-13

The final set of equations, so extracted, is very similar to an FDTD one. For example, the 3D E_z and H_x update expressions at $n + 0.5$ and n time steps respectively, are given by

$$\partial_t E_z = \sum_{ps} C_{ps} \left(\frac{H_y|_{i+ps+0.5,j,k}^{n+0.5}}{\varepsilon_0 \delta x} - \frac{H_x|_{i,j+ps+0.5,k}^{n+0.5}}{\varepsilon_0 \delta y} \right) \tag{5}$$

$$\partial_t H_x = \sum_{ps} C_{ps} \left(\frac{E_y|_{i,j,k+ps+0.5}^{n}}{\mu_0 \delta z} - \frac{E_z|_{i,j+ps+0.5,k}^{n}}{\mu_0 \delta y} \right) \tag{6}$$

where ε_0 and μ_0 are the permittivity and permeability of a vacuum, respectively. It should be pointed out that the scaling functions involved in eqns. 5 and 6 only through coefficients C_{ps}, are calculated once and can be used directly (Table 1). Therefore, the use of scaling functions preserves the simplicity in the algorithm's implementation.

Evidently, more than two H_x, H_y components in eqn. 5 (E_y, E_z in eqn. 6) are used for the computation of $E_z(H_x)$ ones, contrary to the standard FDTD method. Finally, stability of the above hybrid scheme is assured by the correspondingly modified Courant stability condition [7]:

$$c\delta t \le \left(\sum_{ps} |C_{ps}| \sqrt{\frac{1}{\delta x^2} + \frac{1}{\delta y^2} + \frac{1}{\delta z^2}} \right)^{-1} \tag{7}$$

It must also be mentioned that it is the orthogonality and compact support properties of Daubechies' scaling functions that motivated the selection of wavelets instead of other interpolation functions (e.g. polynomials), which seem to be simpler. In fact, the choice of wavelets may be thought of as a systematic construction of interpolation functions, either polynomial or of a more general form, which enables the (otherwise not straightforward) enforcement of orthogonality. Owing to these inherent features of scaling functions, the wavelet–Galerkin method turns out to be a higher-order approximation method and similar to (though more efficient than) the one resulting from the use of polynomials.

3 Hybrid FDTD wavelet–Galerkin technique

Let us consider a waveguide structure (Fig. 1), where conductive arbitrarily placed wedges protrude from the walls. According to the key points of the novel scheme, the computational domain is divided into regions of sharp (in the vicinity of the wedges) and smooth (rest of domain) field transitions, respectively. Each field component (***E*** or ***H***) in the former area is expanded in a set of Daubechies' scaling functions eqn. 1, thus exploiting their ability to precisely simulate highly varying nonlinear phenomena. Hence, a significant improvement in accuracy is achieved, since spatial derivatives in the wavelet–Galerkin update equations are computed via more than two samples, as opposed to the standard Yee formulation. On the other hand, smooth field transition areas, where singnlarities are absent and linear phenomena can be considered, are handled by the more convenient FDTD method.

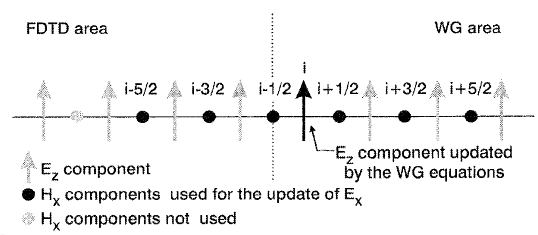

Fig. 2 *Graphical depiction of E_z update near interface of FDTD/WG areas*
H_y component is not shown for simplicity

The coexistence of these two efficient techniques at the interface of the aforementioned regions is straightforward, on condition that spatial and temporal increments are chosen to be equal, a condition that is easily attained. Therefore, each field component of the wavelet–Galerkin area is always computed by $4N - 2$ samples without taking into account the existence of the interface. For example, in eqn. 5, the update of E_z near the interface requires samples from both FDTD and wavelet–Galerkin areas, as shown in Fig. 2. The number of these samples depends on the distance from the interface and from the selected scaling function. Hence, no special interface (transition) conditions are needed. Such a treatment does not cause any inconsistency or instability, avoids unnecessary complications, and maintains the enhanced accuracy.

Nevertheless, regions of sharp transition, discretised by the wavelet–Galerkin method, must be carefully chosen to precisely represent the transient phenomenon. While computation of field components at a cubic cell around the singularity comer by the wavelet–Galerkin equations is adequate to provide reliable results for coarse grids, more components need to be accordingly updated when finer lattices are considered. This can be mainly attributed to the fact that the volume of a cell decreases as the grid becomes finer, and therefore, the sharp transition region demands more cells for its consistent modelling.

Obviously, the innovative algorithm combines all the merits of the two individual methods. Furthermore, the shortcoming of the wavelet–Galerkin method requiring an increased number of time steps in order to reach steady-state, is outbalanced by its use in a confined area. Thus, the overall time needed for the proposed scheme to obtain an accurate solution is notably decreased, compared to the one needed for a pure wavelet–Galerkin algorithm or the standard FDTD method, as revealed by the numerical results.

4 Numerical results

Numerical verification of the proposed technique is performed via diverse waveguides, incorporating both symmetrically and asymmetrically located conductive wedges. By virtue of their infinite length, the above structures are investigated, without loss of generality, in a 2D domain, while an FDTD dense grid simulation plays the role of the reference solution in both test problems. For optimal precision, different scaling functions of varying number of wavelet vanishing moments were implemented, whereas the amount of field components corrected by the FDTD wavelet–Galerkin method was analogously modified.

As a first example, a rectangular waveguide ($5 \times 6mm^2$), characterised by the existence of two symmetrically located perfectly conducting spikes of equal length (2mm), is analysed. Fig. 3 demonstrates the relative error of the TM_1 mode cutoff frequency by means of several FDTD and FDTD wavelet–Galerkin formulations. The promising behaviour of the technique can be easily observed from the essential diminution of lattice size and the total computational time it achieves, as compared to those obtained via dense grid FDTD realisations, given a constant error level. Moreover, it is evident that an increase in the number of

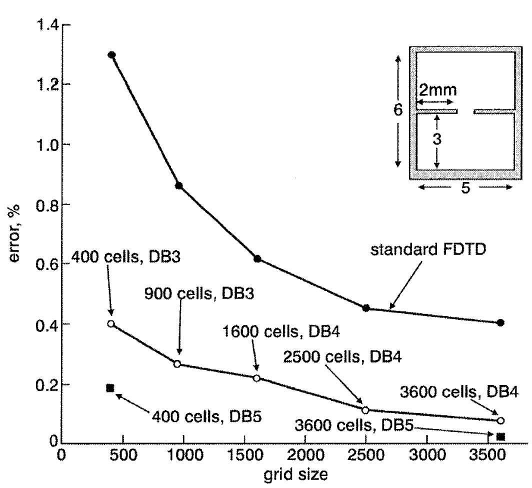

Fig. 3 *Variation of relative error as function of grid size*

vanishing moments, N, provides an additional improvement in the accuracy of the solution. Similar deductions are also experienced by the results of Fig. 4, where the error comparison for the TE resonant frequencies is presented.

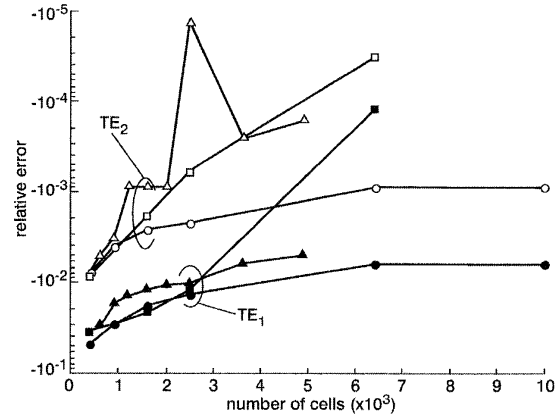

Fig. 4 *Comparison of TE modes for symmetrical wedge case (α = 0°) against grid size*
● ○ FDTD
▲ △ DB3
■ □ DB4

The important issue of the optimum amount of field components corrected by the proposed hybrid scheme is next considered for a 30 × 30 mesh, and results are presented in Table 2. Their close inspection reveals the highly accurate character of our algorithm, along with the specification of a threshold above which no further enhancement is attained for a predetermined grid size.

Table 2: Relative error for symmetrical case (30 × 30)

Components	3	6	8	14	17	24
Error, %	2.34	1.38	0.26	0.31	0.30	0.30

Table 3: Results for asymmetrical case

Technique	TM_1, GHz	Error, %	TM_2, GHz	Error, %	Grid reduction
Hybrid (DB3, 30 × 30)	51.4835	−2.20	73.8884	0.20	96.0%
Hybrid (DB3, 40 × 40)	52.7547	0.20	73.0939	−0.87	92.8%
Hybrid (DB5, 30 × 30)	52.7945	0.28	73.6501	−0.12	96.0%
Hybrid (DB5, 40 × 40)	52.5998	−0.08	73.7295	−0.01	92.8%
FDTD (30 × 30)	52.9136	0.51	73.9361	0.26	96.0%
FDTD (80 × 80)	52.6276	−0.03	73.7931	0.07	71.5%
FDTD (150 × 150)	52.6455	0.0	73.7395	0.0	—
Adaptive FEM	52.5441	−0.19	73.7864	0.06	—

Left wedge: length = 2mm, bottom distance = 3mm
Right wedge: length = 3mm, bottom distance = 2mm

As far as the asymmetrical case is concerned, Table 3 compares the cutoff frequencies of the first two TM modes computed by the FDTD wavelet–Galerkin, the standard FDTD, and adaptive FE [12] methods. Daubechies' scaling function and the mesh size (in cells/dimension) used are indicated in each case. Again, from the results it is concluded that the proposed scheme exhibits the best performance by leading to very satisfactory relative errors, even with a coarse grid. This last remark is proven to be very significant (especially for 3D structures), as the total reduction of the mesh reaches up to 96% of the reference 150 × 150 FDTD domain, thus enabling notable savings in storage and CPU requirements.

Finally, a rectangular waveguide structure having the same characteristics as that of Figs. 3 and 4, but with two perfectly conducting wedges of angle $\alpha = 5°$ and not infinitely thin, is analysed. The inclined edges of the two wedges can no longer be modelled by Cartesian lattices, and hence, a conformal method [13–15] is combined with the techniques used for the analysis of the structure. Results of the relative error in the computed TE resonant frequencies, are presented in Table 4. Obviously, the proposed technique is equally efficient in the degenerate case.

Table 4: Relative error for angled-wedge case

Technique	TE_1	TE_2
Hybrid (DB3, 50 × 50)	−0.533%	−0.089%
Hybrid (DB3, 100 × 100)	−0.0582%	−0.014%
FDTD (50 × 50)	−0.567%	−0.126%
FDTD (150 × 150)	−0.063%	−0.033%

It is noted that the use of DB*N* functions is completely case-independent, and thus, their selection in the above examples was arbitrary.

5 Conclusions

A new hybrid FDTD wavelet–Galerkin technique for the accurate modelling of general wedge-based configurations is presented in this paper. The prominent merits of the algorithm focus on the precise representation of highly varying phenomena, the significant reduction of the grid size, the considerably decreased computational demands, and the simple programming realisation. These advantages, along with its structural versatility, verified by various numerical results addressing complex waveguides, render the proposed scheme a very attractive and powerful tool for the treatment of locally encountered field singularities.

6 References

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