

A Hybrid FDTD–Wavelet-Galerkin Technique for the Numerical Analysis of Field Singularities Inside Waveguides

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Abstract—A novel hybrid Finite Difference Time Domain–Wavelet-Galerkin (FDTD–WG) technique is presented for the accurate representation of electromagnetic field solution in regions of fast field transitions. Its fundamental concept lies on the combination of the robust FDTD method with the Wavelet–Galerkin formulation, which in its turn can efficiently treat highly varying phenomena. The computational domain is, therefore, divided into regions of smooth and fast field variations. Proper determination of these two areas leads to improved simulations and significantly reduced computational burden, turning the proposed scheme into a powerful tool. Numerical results derived from the manipulation of various waveguides, proves the validity and efficacy of the new scheme.

Index Terms—FDTD method, waveguides, wavelet transforms.

I. INTRODUCTION

THE THEORETICAL and numerical representation of electromagnetic field's spatial disturbance due to sharp conductive wedges, has been a point of ongoing research for diverse microwave areas. During the past years, when the applications of microwave and millimeter technology have presented an enormous growth, various techniques have been proposed for the treatment of these singularities, based either on denser FDTD/FEM discretization of the domain so investigated [1] (global solutions), or on the development of hybrid schemes [2]–[4] aiming at the reduction of the overall computational burden (local solutions).

However, numerical simulations indicated that, due to linear field alterations inside each cell (element), the FDTD (FEM) method cannot efficiently approximate sharp transitions and, therefore, it cannot obtain sufficient accuracy unless a very dense global grid or a subgridding technique is used. Obviously, the computational resources required in such cases increase a lot and sometimes become prohibitive, especially in 3-D problems.

Furthermore, despite the good results obtained with fairly coarse grids [3], [4], local solutions, that are implemented mainly in a confined area around the singularity point, turn out to be mathematically more complex contrary to the simplicity characterizing the standard FDTD or FEM codes.

The purpose of this paper is to introduce a novel hybrid numerical algorithm, which, combining the FDTD and the WG methods, can efficiently approximate the field solution in regions of fast variations. A fairly coarse grid has been shown to be adequate for the new scheme to reach a sufficient level of accuracy, while being quite algorithmically elegant and routinely feasible due to the great resemblance of the combined methods. The advantage of the proposed technique stems from the simplicity of the standard FDTD method and the ability of Daubechies' compactly supported scaling functions to represent transient phenomena with a few series terms [5]. As a means of validation of the proposed algorithm's efficiency, various 2-D waveguides containing perfectly conducting wedges are numerically analyzed.

II. WAVELET-GALERKIN FORMULATION

A. The Galerkin Procedure

According to the well known Galerkin Weighted Residuals procedure [6] in order to attain an accurate solution u_n for the differential equation

$$\mathbf{L}u = 0, \quad (1)$$

where \mathbf{L} is a differential operator, the unknown function u is expressed in terms of a set of basis functions φ_i , $i = 1, 2, \dots, n$ by the expansion series

$$u_n = \sum_{i=1}^n \alpha_i \varphi_i, \quad (2)$$

where α_i are calculable coefficients. Equation (2) is then substituted into (1) and the outcome is weighted by means of a proper function g_j over the computational domain, i.e.

$$\sum_{i=1}^n \alpha_i \langle \mathbf{L}\varphi_i, g_j \rangle = 0, \quad j = 1, 2, \dots, n. \quad (3)$$

Since, basis and weighting functions are *a priori* defined, the solution of the $n \times n$ equation system (3) provides the values of coefficients α_i , and, consequently, via (2) an approximate solution of differential equation (1). Obviously, the proper selection of functions φ_i and g_j , is an important issue which decisively affects the accuracy of the procedure.

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B. Scaling Function Families in the Galerkin Procedure

In the WG method [7], [8], as one of the several variations of the Galerkin procedure, various sets of compactly supported scaling functions, recently introduced by Daubechies [9], are chosen as the basis and weighting functions of the procedure. These new function families, characterized by important properties that may simplify a lot the computational procedure, are produced via equation

$$\varphi_k^j(x) = 2^{j2} \varphi(2^j x - k). \quad (4)$$

The generator function is computed via the recursive relation

$$\varphi(x) = \sum_{i=0}^{N-1} h(i) \varphi(2x - i), \quad (5)$$

where h are appropriate sets of coefficients related to each generator. It can be shown that each set of functions $\{\varphi_k\}$ is orthonormal and, therefore, it can be used as a basis for the approximation of an arbitrary function by an expansion series

$$f(x) = \sum_{i=0}^{N-1} \alpha_i \varphi_i(x). \quad (6)$$

This property is called multiresolution analysis and is inherent in all scaling functions. Daubechies families (DBN), utilized in the WG method, are distinguished by their very small support ($[0, N-1]$) which allows them to exactly simulate polynomial functions of degree less than $N/2$.

If function f depends on m variables x_1, x_2, \dots, x_m then it can be expanded in a scaling function series as follows

$$\begin{aligned} f(x_1, \dots, x_m) \\ = \sum_{i_1=0}^{N-1} \dots \sum_{i_m=0}^{N-1} \alpha_{i_1} \dots \alpha_{i_m} \varphi_{i_1}(x_1) \dots \varphi_{i_m}(x_m). \end{aligned} \quad (7)$$

Equation (7) is actually an equivalent expression for (2) in the Galerkin procedure of paragraph II.A.

C. WG Formulation in Electromagnetism

The implementation of the WG algorithm in the area of electromagnetism, requires the expansion of field components in a scaling function series, according to (7) in both time and space domain as

$$F(x, y, t) = \sum_i \sum_j \sum_n F_{i,j}^n \varphi_i(x) \varphi_j(y) h_n(t). \quad (8)$$

In this expansion F represents the electric, \mathbf{E} , and magnetic, \mathbf{H} , field components, $h_n(t)$ is generated by scaling and transition of the well known Haar scaling function and $\varphi_\zeta(\xi)$, $\zeta = i, j$ and $\xi = x, y$, is generated by an orthonormal Daubechies scaling function. Obviously, since each field component is expanded in a scaling function series and such an approximation becomes better as the number of vanishing moments of the corresponding wavelet function increases, the proper selection of $\varphi(\xi)$ affects the accuracy of the method.

Coefficients $F_{i,j}^n$ may be represented in a 2-D time-advancing finite difference grid (Fig. 1). They are located in the same posi-

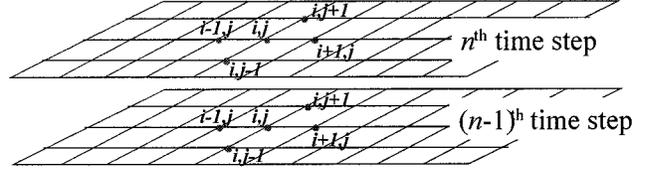


Fig. 1. Representation of the WG coefficients in an FD grid.

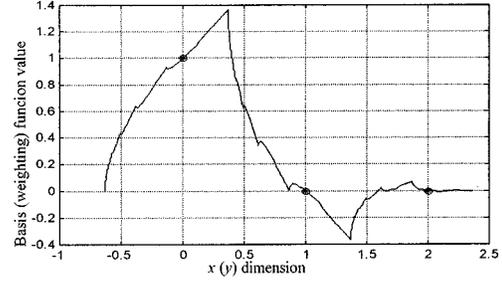


Fig. 2. Shifting of DB2 scaling function by its first moment m_1 [5] results in the equivalence between WG coefficients and the actual electric and magnetic field.

tions as the FDTD field components. It can be also proved that, due to the shifted interpolation property which the Daubechies scaling functions almost satisfy (Fig. 2), each coefficient $F_{i,j}^n$ is the actual electric or magnetic field component of the (i, j) node at the n th time step. It is worth noting that, in Fig. 2, the only node to which a nonzero value of $\varphi_i(x)$ corresponds, is the i th one.

According to the Galerkin Weighted Residuals procedure, (8) is substituted into Maxwell's equations, thus forming a set of six equations containing scaling functions $\varphi(x)$, $\varphi(y)$ and $h(t)$ as well as their derivatives with respect to directions x , y , and time t respectively.

The resulted equations are weighted by proper basis functions, yielding inner products of the form [8]

$$\langle \varphi_k(x), \varphi_{k'}(x) \rangle = \delta_{k,k'} \delta x \quad (9a)$$

$$\langle \varphi_k(x), d_x \varphi_{k'}(x) \rangle = \sum_p r(p) \delta_{k,k'+p} \delta x \quad (9b)$$

Coefficients $r(p)$ depend on the scaling function selected, and are calculated [8], [10] by

$$r(p) = \int_{-\infty}^{+\infty} \varphi(x+p) d_x \varphi(x-0.5) dx \quad (10)$$

Table I presents the coefficients $r(p)$ computed numerically for Daubechies scaling functions, with various numbers of vanishing moments N of the corresponding wavelet function.

III. FDTD–WAVELET–GALERKIN FORMULATION

Let us consider an infinitely long cylindrical waveguide of curvilinear cross-section, which contains two perfectly conducting, asymmetrically located wedges (Fig. 3). The analysis of such a structure via the FDTD method demands

TABLE I
 $r(p)$ COEFFICIENTS FOR VARIOUS SCALING FUNCTIONS

p	DB2	DB3	DB5
0	1.233348	1.291646	1.3160011
1	-0.093853	-0.137122	-0.1642694
2	0.010618	0.028742	0.0503897
3		-0.003466	0.0135287
4		7.98963e-6	0.0022954
5			1.48145e-4
6			-8.60219e-6
7			5.73355e-8
8			2.85768e-13

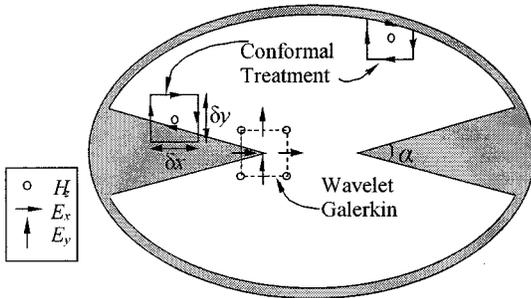


Fig. 3. General curvilinear waveguide structure with singularity regions.

fairly dense grids in the region surrounding the two wedges, as it has already been mentioned. High precision results may also be obtained by the WG method for rather coarse grids, however long computational time is required.

In an effort to overcome the above disadvantages, we propose the division of the computational domain into regions of sharp field transitions (in the vicinity of the wedges) and regions of smooth field transitions (rest of the domain of interest). In the former, the WG method is implemented for the computation of electromagnetic field components, since—due to the multiresolution analysis property—it is able to accurately represent non-linear phenomena without the need for dense grids, unlike the linear standard FDTD. On the contrary the FDTD method is more suitable for the discretization of smooth field variation regions, where singularities are absent, linear phenomena can be considered and the use of WG would be rather pointless.

Therefore, the slow convergence of the WG is outbalanced by its confined use in singularity regions only. Moreover, the overall computational time required for the proposed scheme to reach an accurate solution, is significantly decreased in comparison to the one required for the FDTD or a pure WG algorithm. These remarks are confirmed by the numerical results of the next section.

Additionally, the WG equations are a simple modification of the FDTD ones and therefore, the combination of the two

methods does not require complexity in programming. Specifically, for a TM wave, the WG expressions are

$$H_x|_{i,j}^{n+0.5} = H_x|_{i,j}^{n-0.5} - \sum_p r(p) \frac{\delta t}{\mu_0 \delta y} E_z|_{i,j+p+0.5}^n$$

$$H_y|_{i,j}^{n+0.5} = H_y|_{i,j}^{n-0.5} + \sum_p r(p) \frac{\delta t}{\mu_0 \delta x} E_z|_{i+p+0.5,j}^n$$

$$E_z|_{i,j}^{n+1} = E_z|_{i,j}^n + \sum_p r(p) \cdot \left(\frac{\delta t}{\epsilon_0 \delta x} H_y|_{i+p+0.5,j}^{n+0.5} - \frac{\delta t}{\epsilon_0 \delta y} H_x|_{i,j+p+0.5}^{n+0.5} \right).$$

The great resemblance of the WG and the FDTD methods is also seen from the fact that for $r(-1) = -1$ and $r(0) = 1$ the former coincides with the latter.

However, regions of sharp transitions, discretized by the WG method, must be carefully chosen in order to accurately represent the transient phenomenon. Due to the decrease of a single cell's area as the grid is refined, more cells are needed for the consistent modeling of the region. Thus, more field components must be corrected by the WG equations.

Moreover, for the precise modeling of the system's curved walls and inclined edges, the proposed methodology implements contours, which enable subcell conformal simulation of the interface. Therefore, the rigorous enforcement of the zero tangential electric field condition is consistently fulfilled and the poor resolution of the staircase approximation is readily avoided. It must be noted that this conformal scheme is a versatile tool in dealing with field singularities near wedges of arbitrary angle, located inside general non-Cartesian structures, since it introduces increased accuracy to the computational procedure.

Finally, the stability of the hybrid technique is assured by a modification of the Courant stability criterion

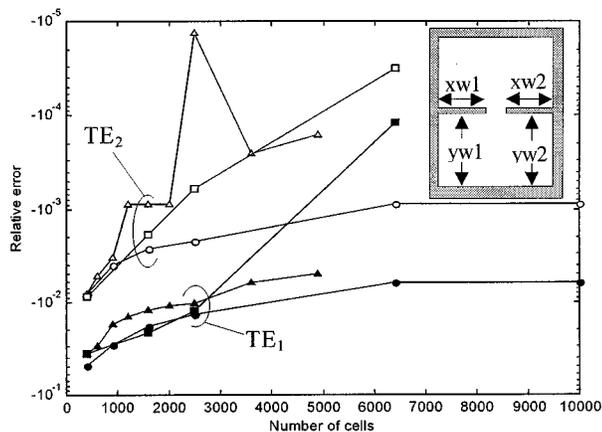
$$\delta t \leq \left(c \sum_p |r(p)| \sqrt{\delta x^{-2} + \delta y^{-2}} \right)^{-1}, \quad (11)$$

since for the same grid size, the WG requires smaller time step than the standard FDTD method.

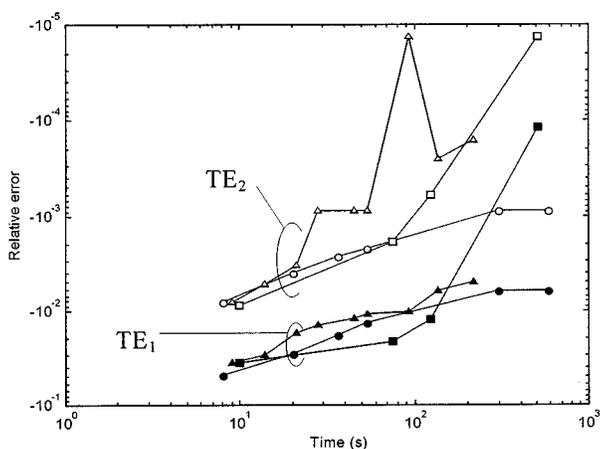
IV. NUMERICAL RESULTS

The new hybrid algorithm is validated by its implementation in the computation of the resonant frequencies of various waveguide structures that belong to the general case of Fig. 3. The results of the proposed technique are each time compared with those of the FDTD method.

At first, a symmetrical rectangular waveguide ($5 \times 6 \text{ mm}^2$) is examined, containing two wedges of equal length (2 mm) which have the degenerate form of an infinitely thin, perfectly conducting spike ($\alpha = 0^\circ$). For the problem's manipulation via the new technique, two scaling functions are used, Daubechies' scaling function with two (DB2) and four (DB4) vanishing moments, respectively. Fig. 4(a) and (b) illustrate the relative error produced by each of the two approaches (FDTD-WG, FDTD) with respect to the number of cells used and the computational time respectively. It is observed that the FDTD method requires



(a)



(b)

Fig. 4. (a) Error comparison for the first two TE modes of a symmetrical wedge case ($\alpha = 0^\circ$) for various grids: FDTD(circles), DB2(triangles), DB4(square), (b) Error comparison for the first two TE modes of a symmetrical wedge case ($\alpha = 0^\circ$) versus time: FDTD(circles), DB2(triangles), DB4(squares).

twice as much time to attain the same level of accuracy as the proposed technique. Furthermore, the choice of scaling function DB4 slightly improves the accuracy of the technique without significantly increasing the computational time. Finally, field distribution in a confined area around the wedge is shown in Fig. 5.

The proposed scheme was also used for the computation of the resonant frequencies of the same structure but with asymmetrically located wedges ($yw1 = 3$ mm, $yw2 = 2$ mm from the bottom side, $xw1 = 2$ mm from the left side and $xw2 = 3$ mm from the right side). In Table II, a comparison of the two first TM cutoff frequencies computed by the FDTD–WG technique, the standard FDTD method and a higher order FEM [11], is presented, where the results obtained with a fine 150×150 FDTD grid are taken as a reference (shaded row). This comparison proves the efficiency of the proposed technique, since its accuracy is of the same order as that of the FDTD algorithm despite the fact that the grid size is reduced by 95%. Furthermore, the computational time is reduced by approximately 50%.

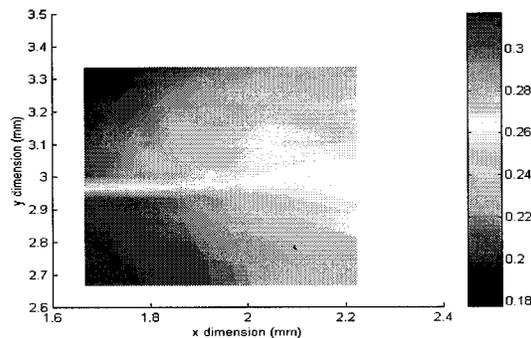


Fig. 5. Field distribution produced by the FDTD–WG technique inside the region of fast field variations (30×30 grid, DB2).

TABLE II
RESULTS OBTAINED FOR THE ASYMMETRICAL WAVEGUIDE PROBLEM

Technique Implemented	TM ₁ (GHz)	Error (%)	TM ₂ (GHz)	Error (%)	Cell Reduction
Hybrid (DB3, 30x30)	51.4835	-2.20	73.8884	0.20	96.0 %
Hybrid (DB3, 40x40)	52.7547	0.20	73.0939	-0.87	92.8 %
Hybrid (DB5, 30x30)	52.7945	0.28	73.6501	-0.12	96.0 %
Hybrid (DB5, 40x40)	52.5998	-0.08	73.7295	-0.01	92.8 %
FDTD (30x30)	52.9136	0.51	73.9361	0.26	96.0 %
FDTD (80x80)	52.6276	-0.03	73.7931	0.07	71.5 %
FDTD (150x150)	52.6455	-	73.7395	-	-
Higher-order FEM[11]	52.5441	-0.19	73.7864	0.06	-

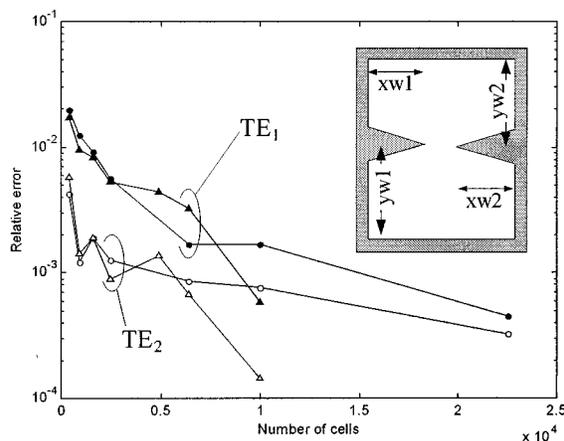


Fig. 6. Error comparison for the first two TE modes of a symmetrical wedge case ($\alpha = 5^\circ$) for various grids: FDTD(circles), DB2(triangles).

Finally, a rectangular waveguide having the same structure as that of Figs. 4 and 5, but with two perfectly conducting wedges of angle $\alpha \neq 0^\circ$ and not infinitely thin, was analyzed. The inclined edges can no longer be modeled by Cartesian lattices, as shown in the previous section, and, therefore, a conformal method is combined with the techniques used for the analysis of the structure. The comparative results of the relative error in the computed resonant frequencies, are illustrated in Fig. 6. Obviously, the proposed technique is equally efficient as in the degenerate case.

It is noted that the results obtained by the implementation of the FDTD method, for an extremely dense grid, are considered as reference in both cases.

V. CONCLUSIONS

A novel hybrid technique appropriate for the treatment of field singularities is introduced. It is based on the division of the computational domain into two kinds of regions according to the field alteration rate. The Wavelet–Galerkin algorithm is locally implemented in the region of sharp variations, whereas the FDTD method is utilized in the rest of the computational domain. The adjustment of the two methods is quite simple, since the Wavelet–Galerkin equations do not essentially differ from the FDTD ones. The proposed scheme's efficiency is proved via its performance comparison with that of the FDTD method in the numerical manipulation of various waveguides.

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